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Injectivity and projectivity of supercuspidals

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1. Projectivity of supercuspidal irreducibles

This important result must be understood accurately: it is *not* asserted that supercuspidal irreducibles are projective in the *whole* category of smooth representations of G, but only in the smaller category of representations with a *fixed 'central character'*.

Let $\mathcal{H}_{\chi^{-1}}$ be the Hecke algebra of locally constant $\mathbf{C}$-valued functions on G which are compactly-supported modulo Z, and which are (Z, χ^{-1})-equivariant in the sense that

$$f(zg) = \chi^{-1}(z) f(g)$$

Let π be an irreducible supercuspidal representation of G, i.e., whose coefficient functions are compactly-supported modulo Z. Let χ be the central character of π. Then the coefficient functions $c^{\pi}_{v,\lambda}$ of π are in $\mathcal{H}_\chi$.

Proposition: In the category $\mathcal{C}$ of smooth representations (ρ, X) of G with 'central character' χ, irreducible supercuspidal representations (π, V) (with central character χ are *projective*. That is, given a surjection

$$\varphi : X \to V$$

in $\mathcal{C}$, there is a ('section') $\sigma : V \to X$ so that

$$\varphi \circ \sigma = 1_V$$

Proof: We need to use the facts, proven earlier, that irreducible supercuspidals π are admissible, and that their smooth duals $\check{\pi}$ are likewise admissible and supercuspidal. For example, it follows that $\check{\check{\pi}} \approx \pi$.

Let $\mathcal{A}$ be the subalgebra of $\mathcal{H}_{\chi^{-1}}$ generated by the coefficient functions of the smooth dual $\check{\pi}$ of π. From the definition of supercuspidal, these coefficient functions are in $\mathcal{H}_{\chi^{-1}}$. Indeed, the space of all such coefficient functions is

$$\mathcal{A} \approx \check{\pi} \otimes \pi$$

as $G \times G$-space, where the first G acts by right regular representation and the second by left regular. Let $\mathcal{K} \subset \mathcal{H}_{\chi^{-1}}$ be the intersection of the kernels of all the maps

$$\eta \to \pi(\eta) v$$

for $v \in V$. Then it is immediate that

$$\mathcal{H}_{\chi^{-1}} = \mathcal{A} \oplus \mathcal{K}$$

Fix non-zero $v_o \in \pi$, and let $x_o \in X$ be an element so that $\varphi(x_o) = v_o$. Take $\lambda_o \in \check{\pi}$ so that $\lambda_o(v_o) = 1$. Then for

$$\eta = \lambda_o \otimes v \in \check{\pi} \otimes \pi \subset \mathcal{A} \subset \mathcal{H}$$

define

$$\sigma(\pi(\eta) v_o) = \rho(\eta) x_o$$

If $(\lambda_o \otimes v)v_o = 0$ then $v = 0$, so this is indeed a well-defined map. The assumption that π is supercuspidal is what allows us to make such a choice of $\eta \in \mathcal{H}_{\chi^{-1}}$ to obtain arbitrary elements of π from a given non-zero vector.

Then the design of the definition of σ makes the proof that σ gives a one-sided inverse to φ easy:

$$\varphi(\sigma(\pi(\lambda_o \otimes v)v_o))) = \varphi(\rho(\lambda_o \otimes v)x_o) =$$

$$= (\lambda_o \otimes v)\varphi(x_o) = (\lambda_o \otimes v)v_o = \lambda_o(v_o)v = v$$

where we use the fact that φ is a G-morphism, so commutes with the action of the Hecke algebra. ♣

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2. Injectivity of supercuspidal irreducibles

Again: it is *not* asserted that supercuspidal irreducibles are injective in the *whole* category of smooth representations of G, but only in the smaller category of representations with a *fixed 'central character'*.

Corollary: A supercuspidal irreducible (π, V) with 'central character' χ is *injective* in the category $\mathcal{C}(\chi)$ of smooth representations with 'central character' χ. That is, for an injection

$$\varphi : V \to X$$

in $\mathcal{C}(\chi)$, there is $\sigma : X \to V$ so that $\sigma \circ \varphi = 1_V$.

Proof: We obtain a natural surjective dual map

$$\check{\varphi} : \check{X} \to \check{V}$$

where the surjectivity follows from the (trivial) 'Hahn-Banach' theorem relevant here. Since $\check{\pi}$ is supercuspidal irreducible in $\mathcal{C}(\chi^{-1})$, it is injective in $\mathcal{C}(\chi^{-1})$, so there is $\tau : \check{\pi} \to \check{V}$ so that $\check{\varphi} \circ \tau = 1_{\check{V}}$. Dualizing again, using the fact that $\check{\check{\pi}} \approx \pi$ because of *admissibility*, we obtain

$$\check{\tau} : \check{\check{X}} \to V$$

so that

$$\check{\tau} \circ \varphi = \check{1}_{\check{V}} = 1_{\check{\check{V}}} = 1_V$$

So $\check{\tau}$ restricted to $X \subset \check{\check{X}}$ is the desired one-sided inverse to φ. ♣